

Trans-Planckian fluctuations and the stability of quantum mechanics

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We present arguments suggesting that deviations from the Born probability rule could be generated for trans-Planckian field modes during inflation. Such deviations are theoretically possible in the de Broglie-Bohm pilot-wave formulation of quantum mechanics, according to which the Born rule describes a state of statistical equilibrium. We suggest that a stable equilibrium state can exist only in restricted conditions: on a classical background spacetime that is globally hyperbolic or in a mild quantum-gravity regime in which there is an effective Schrödinger equation with a well-defined time parameter. These arguments suggest that quantum equilibrium will be unstable at the Planck scale. We construct a model in which quantum nonequilibrium is generated by a time-dependent regulator for pilot-wave dynamics, where the regulator is introduced to eliminate phase singularities. Applying our model to trans-Planckian modes that exit the Planck radius, we calculate the corrected primordial power spectrum and show that it displays a power excess (above a critical wavenumber). We briefly consider how our proposals could be tested by measurements of the cosmic microwave background.

1 Introduction

According to our current understanding, the observed anisotropies in the cosmic microwave background (CMB) were seeded by primordial quantum fluctuations that were generated during an inflationary expansion [1, 2, 3, 4]. Precision measurements of the CMB then allow us to test fundamental physics at very early times and at very short distances. It has been argued that trans-Planckian field modes – modes with early physical wavelengths λ_{phys} smaller than the Planck length l_{P} – are likely to contribute significantly to the inflationary spectrum [5, 6]. In this case, inflationary cosmology would allow us to probe physics at the Planck scale and beyond (for a review see ref. [7]).

The physical meaning of sub-Planckian lengthscales can of course be questioned, and it may well be that an account in terms of modes with $\lambda_{\text{phys}} < l_{\text{P}}$ is only an effective description. Seen in this way, the ‘trans-Planckian problem’ of inflationary cosmology provides an opportunity to probe physics at the interface of quantum theory and gravitation [7], an area concerning which there is as yet little consensus and no well-established theory.

Most approaches to quantum gravity apply the standard rules of quantum mechanics to the gravitational field [8, 9, 10, 11, 12, 13]. Despite many successes, conceptual problems remain (see for example refs. [14, 15, 16]).

In this paper we suggest that standard approaches to quantum gravity are based on an implicit assumption that could turn out to be incorrect: that the quantum-theoretical Born probability rule still holds at the Planck scale. We suggest that this rule is relevant only in restricted conditions: on a classical background spacetime that is globally hyperbolic, or in a mild quantum-gravity regime with an effective time-dependent Schrödinger equation. In these regimes one may define a conserved quantum current in configuration space and apply the Born rule in the usual way. But more generally – for example in the spacetime associated with the formation and complete evaporation of a black hole, or in the deep quantum-gravity regime – we suggest that there simply is no Born rule and that more general probabilities are possible.

To make sense of this suggestion requires a formulation of quantum mechanics in which the Born rule is not an axiom. Such a formulation is provided by the pilot-wave theory of de Broglie and Bohm [17, 18, 19, 20, 21]. In pilot-wave theory, the Born rule has a dynamical origin and is roughly analogous to thermal equilibrium in classical physics [22, 23, 24, 25, 26, 27, 28, 29]. While the Born rule and its empirical predictions are fully recovered in equilibrium [19, 20], deviations from equilibrium and from the Born rule are theoretically possible [22, 30, 23, 31, 24, 32, 33, 34, 35, 36, 37, 38].

If such deviations existed they would generate new physics beyond the domain of conventional quantum theory. This would include nonlocal signalling [30], which is causally consistent if one adopts an underlying preferred foliation of spacetime [39]. It would also be possible to perform ‘subquantum’ measurements that violate the uncertainty principle and other standard quantum constraints [32, 38]. On this view, quantum physics is an effective theory of an equilibrium state and a much wider nonequilibrium physics can exist at least in

principle.

Such wider physics could have existed in the very early universe before relaxation to equilibrium took place [22, 30, 23, 31]. It has been shown that early quantum nonequilibrium can leave observable traces today, in particular in the cosmic microwave background (CMB) [33, 34, 35, 36, 40, 41, 42] (and perhaps in relic systems that decoupled at very early times [24, 33, 34, 43]). In a cosmology with a radiation-dominated pre-inflationary phase [44, 45, 46, 47, 48], it is natural to expect a large-scale power deficit in the inflationary spectrum induced by a suppression or retardation of early relaxation at long (super-Hubble) wavelengths [33, 34, 36, 40, 41]. With appropriate cosmological parameters the expected deficit is roughly consistent with that observed in the CMB by the *Planck* satellite [49, 40, 41]. While the observed deficit may well be caused by some other more conventional effect, the fact remains that inflationary cosmology provides us with a new and powerful empirical window onto the Born rule in the very early universe.

In this paper we consider a different possible origin for early violations of the Born rule. It will be suggested that quantum nonequilibrium can be *created* from an earlier equilibrium state by novel processes taking place at the Planck scale. In addition to arguments that a stable equilibrium state may exist only in restricted gravitational conditions (on a globally hyperbolic spacetime or in a mild quantum-gravity regime with an effective Schrödinger equation), we also point out that the structure of pilot-wave dynamics itself suggests a natural mechanism for the creation of nonequilibrium at short lengthscales in configuration space.

Like classical general relativity, pilot-wave theory suffers from singularities. Specifically, the de Broglie velocity field can diverge at nodes of the wave function. To eliminate these ‘phase singularities’, the theory must be regularised – an elementary point that is usually ignored since the singularities are of measure zero (with respect to the standard volume measure). But by including a regularisation, and allowing it to become time-dependent, one may readily construct a simple modification of pilot-wave dynamics in which nonequilibrium is generated from a prior equilibrium state. This is not intended to be a fundamental theory, but only an effective or phenomenological model of possible novel physics at the Planck scale – in the same spirit in which regularisation procedures in quantum field theory are not regarded as fundamental but only as effective accounts of some unknown physics at very short distances.

By applying our modified pilot-wave dynamics to the inflationary Bunch-Davies vacuum, we obtain a model in which quantum nonequilibrium is created for trans-Planckian modes as they exit the Planck radius. For a mode with wavenumber k , nonequilibrium may be quantified by a function $\xi(k)$ equal to the ratio of the nonequilibrium variance to the equilibrium (Born-rule) variance. For a given regulator, it is possible to calculate $\xi(k)$ and so obtain the modified primordial power spectrum, which is equal to the standard spectrum corrected by the factor $\xi(k)$. Our model predicts a power excess ($\xi > 1$) with a particular dependence on k . Corrections to the standard spectrum set in above a critical wavenumber k_c , set by the comoving wavelength $\lambda_c = 2\pi/k_c$ below which early

inflationary modes were sub-Planckian. At least in principle, best-fits to the available CMB data could provide constraints on this kind of model, though we do not attempt to perform such fits here.

A number of authors have proposed phenomenological or effective accounts of trans-Planckian modifications of quantum field theory (such as modified dispersion relations), with the aim of obtaining constraints from CMB data [7]. Such an approach may be taken, pending the development of a deeper theory. In a similar spirit, we propose trans-Planckian modifications of quantum mechanics itself. It is important to bear in mind that different formulations of the same physics – at the level at which the laws are currently known – are likely to suggest different generalisations, modifications or extensions into a new physical domain where the laws are as yet unknown. Thus, if one approaches the trans-Planckian domain from the perspective of standard quantum field theory, it is natural to consider modifications of dispersion relations, commutation relations, and other elementary field-theoretical properties. From the perspective of the pilot-wave formulation of quantum field theory, it is natural to consider that the Born rule may be modified – a proposal that is not conceivable in standard quantum field theory but which is conceptually clear in pilot-wave theory.

In Section 2 we consider our three separate arguments suggesting that the Born rule could be unstable at the Planck scale. The first two arguments point out that the usual derivations of a conserved probability current depend either on the existence of a background globally-hyperbolic spacetime or on the existence of an effective Schrödinger equation, and that both requirements can arguably be broken by gravitational effects. The third argument highlights the existence of phase singularities in pilot-wave dynamics, which require regularisation at short distances in configuration space. In Section 3 we review the pilot-wave theory of a scalar field on expanding flat space, in particular for the Bunch-Davies vacuum in de Sitter space. In Section 4 we discuss our model for quantum instability, in which deviations from the Born rule can be generated by a time-dependent regularisation of pilot-wave dynamics. We apply this model to the Bunch-Davies vacuum and we calculate the primordial power spectrum, which is corrected with respect to the usual result by a factor $\xi(k)$ that generally exceeds unity. These effects can exist for modes above a certain wavenumber k_c , which underwent a Planck radius exit some time during the inflationary phase. In Section 5 we provide a simple estimate of k_c and discuss how the observability of the relevant region of k -space depends on the values of basic cosmological parameters. In Section 6 we summarise our conclusions and suggestions for future work.

2 Arguments for quantum instability at the Planck scale

In non-gravitational physics, quantum equilibrium appears to be stable in the sense that it is preserved in time under standard operations and interactions.

The Born rule continues to hold, for instance, in high-energy collisions (as probed by scattering cross-sections). In pilot-wave theory this stability is a simple consequence of the dynamics, which evolves an initial equilibrium distribution to a final one.

A general system in pilot-wave theory has an evolving configuration $q(t)$ as well as a wave function $\psi(q, t)$. Here t is an external time parameter. In high-energy physics, t is the time associated with an underlying preferred rest frame – or preferred foliation of spacetime by spacelike hypersurfaces – and q is the configuration of the fields and particles on 3-space. The Schrödinger equation

$$i \frac{\partial \psi}{\partial t} = \hat{H} \psi \quad (1)$$

(with $\hbar = 1$) has an associated current $j = j[\psi] = j(q, t)$ in configuration space, obeying a continuity equation

$$\frac{\partial |\psi|^2}{\partial t} + \partial_q \cdot j = 0 \quad (2)$$

(where ∂_q is a generalised gradient). We may then define a pilot-wave dynamics for the system. Introducing a configuration-space velocity field

$$v(q, t) \equiv \frac{j(q, t)}{|\psi(q, t)|^2} , \quad (3)$$

we may write the de Broglie equation of motion

$$\frac{dq}{dt} = v(q, t) \quad (4)$$

for actual trajectories $q(t)$ in configuration space. Such a velocity field v exists whenever \hat{H} is given by a differential operator, where the form of v (and j) is determined by the form of \hat{H} [50]. For standard Hamiltonians that are quadratic in the canonical momenta, the components v_a of v are proportional to the components of the phase gradient:

$$v_a \propto \partial_{q_a} S = \text{Im} \left(\frac{\partial_{q_a} \psi}{\psi} \right) . \quad (5)$$

Note that the ‘pilot wave’ ψ is a complex-valued field on configuration space that guides the motion of a single system; it has no intrinsic connection with probability.

For an ensemble of systems with the same wave function $\psi(q, t)$ we may consider the time evolution of an arbitrary distribution $\rho(q, t)$ of configurations $q(t)$, where by construction $\rho(q, t)$ will obey the continuity equation

$$\frac{\partial \rho}{\partial t} + \partial_q \cdot (\rho v) = 0 . \quad (6)$$

This is the same as the continuity equation (2) for $|\psi|^2$. It follows that an initial distribution $\rho(q, t_i) = |\psi(q, t_i)|^2$ at time t_i evolves into a final distribution

$$\rho(q, t) = |\psi(q, t)|^2 \quad (7)$$

at time t .

One may also consider the time evolution of the ratio

$$f \equiv \frac{\rho}{|\psi|^2} \quad (8)$$

along trajectories. From (6) and (2) it follows that

$$\frac{df}{dt} = 0, \quad (9)$$

where $d/dt = \partial/\partial t + v \cdot \partial_q$ is the time derivative along a trajectory.

In the state (7) of ‘quantum equilibrium’ we obtain agreement with the empirical predictions of quantum theory [19, 20]. On the other hand, for a nonequilibrium ensemble ($\rho(q, t) \neq |\psi(q, t)|^2$) the statistical predictions generally disagree with those of quantum theory [22, 30, 23, 31, 24, 32, 33, 34, 35, 36, 37, 38].

In pilot-wave dynamics the quantum equilibrium state (7) is stable in two senses: firstly, an initial equilibrium state remains in equilibrium; and secondly, perturbations away from equilibrium tend to relax.¹ The relaxation process is roughly analogous to thermal relaxation and may be quantified by the decrease of an H -function

$$H = \int dq \, \rho \ln(\rho/|\psi|^2) \quad (10)$$

(on a coarse-grained level) [22, 23, 24, 25, 26, 27, 28, 29].² Such relaxation presumably took place in the very early universe [22, 30, 23, 31].

It is the first sense of stability that concerns us here. By the above simple reasoning, the existence of a quantum equilibrium state that is preserved by the velocity field (3) may be readily established for any system that obeys a Schrödinger equation with an associated conserved current j . For example, for a bosonic scalar field ϕ on Minkowski spacetime we may write quantum field theory in the functional Schrödinger picture, with a wave functional $\Psi[\phi, t]$, and we may assume that the velocity $\partial\phi(\mathbf{x}, t)/\partial t$ of the actual field configuration $\phi(\mathbf{x}, t)$ is given by the functional derivative $\delta S/\delta\phi(\mathbf{x})$ where $S = \text{Im} \ln \psi$ is the phase of Ψ . Fermions may be described by a Dirac-sea picture, with particle trajectories determined by a many-body Dirac wave function [52, 53, 54]. These constructions require a preferred time parameter t with respect to which non-local effects (arising from the equation of motion (4) for entangled quantum states) occur instantaneously. For ensembles of fields or particles in quantum

¹As shown in ref. [51], stability in this second sense does not hold for Bohm’s 1952 second-order reformulation of de Broglie’s original 1927 first-order dynamics.

²Coarse-graining is required, because of the fine-grained conservation (9) of the ratio f . As in the analogous classical case, it must be assumed that the initial state has no fine-grained micro-structure [22, 23, 24, 25].

equilibrium, we recover standard quantum field theory (and hence an effective Lorentz invariance) [55, 42].

In the absence of gravitation, then, the existence of a quantum equilibrium state is a trivial consequence of the structure of pilot-wave dynamics. In the presence of gravitation, however, the situation is not so clear. We shall now present arguments suggesting that quantum equilibrium may in fact be gravitationally unstable.

2.1 Globally-hyperbolic spacetime and the existence of a quantum equilibrium state

The existence of a quantum equilibrium state may be readily established on a classical curved spacetime background that is globally hyperbolic [56].

Such a spacetime may always be foliated (in general nonuniquely) by space-like hypersurfaces $\Sigma(t)$ that are labelled by a global time function t . The space-time line element $d\tau^2 = {}^{(4)}g_{\mu\nu}dx^\mu dx^\nu$ with 4-metric ${}^{(4)}g_{\mu\nu}$ may then be written in the standard 3+1 form

$$d\tau^2 = (N^2 - N_i N^i) dt^2 - 2N_i dx^i dt - g_{ij} dx^i dx^j, \quad (11)$$

where N is the lapse function, N^i is the shift vector and g_{ij} is the 3-metric on $\Sigma(t)$. We may set $N^i = 0$ (for as long as the lines $x^i = \text{const.}$, chosen to be normal to the slices Σ , do not encounter singularities).

For example, for a massless and minimally-coupled real scalar field ϕ with Lagrangian density

$$\mathcal{L} = \frac{1}{2} \sqrt{-{}^{(4)}g} {}^{(4)}g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \quad (12)$$

we have a canonical momentum density $\pi = \partial\mathcal{L}/\partial\dot{\phi} = (\sqrt{g}/N)\dot{\phi}$ (where ${}^{(4)}g = \det g_{\mu\nu}$ and $g = \det g_{ij}$) and a classical Hamiltonian

$$H = \int d^3x \frac{1}{2} N \sqrt{g} \left(\frac{1}{g} \pi^2 + g^{ij} \partial_i \phi \partial_j \phi \right). \quad (13)$$

The wave functional $\Psi[\phi, t]$ then satisfies the Schrödinger equation³

$$i \frac{\partial \Psi}{\partial t} = \int d^3x \frac{1}{2} N \sqrt{g} \left(-\frac{1}{g} \frac{\delta^2}{\delta \phi^2} + g^{ij} \partial_i \phi \partial_j \phi \right) \Psi. \quad (14)$$

This implies a continuity equation

$$\frac{\partial |\Psi|^2}{\partial t} + \int d^3x \frac{\delta}{\delta \phi} \left(|\Psi|^2 \frac{N}{\sqrt{g}} \frac{\delta S}{\delta \phi} \right) = 0 \quad (15)$$

³As usual in this context, we implicitly assume some form of regularisation – such as an analytical continuation of the number of space dimensions away from 3 (see, for example, ref. [57]).

with a current $j = |\Psi|^2 (N/\sqrt{g})\delta S/\delta\phi$ and a de Broglie velocity field

$$\frac{\partial\phi}{\partial t} = \frac{N}{\sqrt{g}} \frac{\delta S}{\delta\phi}, \quad (16)$$

where $\Psi = |\Psi| e^{iS}$ [56, 42].

The field velocity (16) at a point x^i on $\Sigma(t)$ will depend instantaneously (with respect to t) on field values at distant points $(x')^i \neq x^i$ – if Ψ is entangled with respect to the fields at those points. For a nonequilibrium ensemble, a change in the local Hamiltonian at $(x')^i$ will in general instantaneously affect the time evolution of the marginal distribution at x^i , yielding nonlocal signals from $(x')^i$ to x^i (whereas in equilibrium such signals will vanish) [30]. To ensure physical consistency, we assume that the theory has been constructed using a preferred foliation associated with a specific lapse function $N(x^i, t)$ [39].

By construction, an arbitrary distribution $P[\phi, t]$ will satisfy the same continuity equation:

$$\frac{\partial P}{\partial t} + \int d^3x \frac{\delta}{\delta\phi} \left(P \frac{N}{\sqrt{g}} \frac{\delta S}{\delta\phi} \right) = 0. \quad (17)$$

It then follows as usual that $P[\phi, t] = |\Psi[\phi, t]|^2$ is an equilibrium state: if it holds at some initial time it will hold at all times. Thus there is a quantum equilibrium state even in the presence of gravitation – at least for a classical curved spacetime background that is globally hyperbolic.

It is however difficult to see how a comparable construction could be given for a background spacetime that is not globally hyperbolic – such as the spacetime generated by the formation and (complete) evaporation of a black hole [58]. Even standard quantum field theory on curved spacetime relies on the assumption that the spacetime is globally hyperbolic. The usual quantisation procedure imposes canonical commutation relations on a Cauchy surface, so that the wave equation has a well-posed initial value formulation (see for example ref. [59]). In effect the standard theory depends on the quantisation of a well-posed Hamiltonian dynamics for classical fields, and is therefore strictly speaking applicable only to globally-hyperbolic spacetimes. An algebraic approach to quantum field theory on non-globally-hyperbolic spacetimes has been developed and applied to simple, flat (two-dimensional) examples [60]. In this construction, the algebraically-defined quantum state must be specified on the entire spacetime with boundary conditions at naked singularities. It is unclear if this approach could be given a de Broglie-Bohm formulation.

Existing pilot-wave theories require a preferred hypersurface along which nonlocality acts [39]. Even in flat spacetime, attempts to write down a fundamentally Lorentz-invariant pilot-wave theory run into problems associated with nonlocality: both the dynamics and the quantum equilibrium distribution must be defined on a preferred spacelike hypersurface [61, 62, 63, 64].⁴ In the absence

⁴Ref. [65] considers a new velocity law (replacing (4)) involving a non-integrable time-like vector field n^μ , with the aim of formulating a fundamentally Lorentz-invariant pilot-wave theory. However, because the model has no well-defined foliation (or global time function)

of a Cauchy hypersurface, we may expect a fundamental difficulty in defining a quantum equilibrium state for a nonlocal hidden-variables theory.

The de Broglie-Bohm construction depends on the existence of a local quantum current in configuration space, and there seems to be no reason why such a current would exist for a non-globally hyperbolic spacetime. Pending a demonstration to the contrary, one may consider the possibility that such a current does not exist. It has in fact been suggested that there is no well-defined state of quantum equilibrium for a non-globally hyperbolic spacetime and that the formation and complete evaporation of a black hole could generate quantum nonequilibrium from a prior equilibrium state [56, 33]. If such effects existed, outgoing Hawking radiation would be in a state of quantum nonequilibrium and could therefore carry more information than ordinary radiation – potentially offering a new approach to the (controversial) question of information loss in black holes [56, 33].

2.2 Possible non-existence of an equilibrium state in quantum gravity

The existence of a quantum equilibrium state is difficult to establish in canonical quantum gravity because of the absence of a general time-dependent Schrödinger equation with an associated probability current. Here we suggest that it may in fact be a mistake to assume that there exists a state of quantum equilibrium at the Planck scale.

Canonical quantum gravity begins with the Einstein-Hilbert action

$$I = \int d^4x \, (-{}^{(4)}g)^{1/2} {}^{(4)}R$$

(units $G = 1/16\pi$). Employing the standard 3+1 splitting (11), the arbitrariness of the lapse function N implies the Wheeler-DeWitt equation [8]

$$\left(-G_{ijkl} \frac{\delta^2}{\delta g_{ij} \delta g_{kl}} - g^{1/2} R \right) \Psi = 0 , \quad (18)$$

where $G_{ijkl} = \frac{1}{2}g^{-1/2}(g_{ik}g_{jl} + g_{il}g_{jk} - g_{ij}g_{kl})$ is the superspace metric, R is the 3-scalar curvature, and we employ the metric representation with a wave functional $\Psi = \Psi[g_{ij}]$. (Further constraints expressing spatial diffeomorphism invariance on Σ read $(\delta\Psi/\delta g_{ij})_{;j} = 0$, where $;$ is the 3-covariant derivative.)

Many applications of (18) to quantum cosmology make use of a semiclassical WKB approach, where one writes $\Psi = |\Psi| e^{iS}$ – with $|\Psi|$ varying slowly with g_{ij} – and assumes approximately classical trajectories for g_{ij} given by $p^{ij} = \delta S/\delta g_{ij}$, where p^{ij} is the momentum density canonically-conjugate to g_{ij} . Using the canonical relation between p^{ij} and \dot{g}_{ij} , we then have an equation of motion

$$\frac{\partial g_{ij}}{\partial t} = 2NG_{ijkl} \frac{\delta S}{\delta g_{kl}} + N_{i;j} + N_{j;i} . \quad (19)$$

the equilibrium state is ill-defined – except when n^μ happens to be integrable and determines a preferred foliation.

The trajectories defined by (19) are in effect de Broglie-Bohm trajectories, for the special case of a WKB wave functional. A number of authors have proposed a pilot-wave formulation of quantum gravity based on (18) and (19) – with (19) assumed to be valid for any solution $\Psi[g_{ij}]$ of (18) [66, 21, 67]. The resulting theory has been extensively applied to quantum cosmology (for a review see ref. [68]).

At the level of individual systems, an important question concerns the consistency of the pilot-wave dynamics defined by (18) and (19). The lapse and shift functions N and N^i are arbitrary, so any change in these should not affect the resulting 4-geometry traced out by the evolution of the 3-geometry. Otherwise the initial-value problem would be ill-posed. Shtanov [67] gave an example for which it appeared that the predicted 4-geometry would depend on the arbitrary choice of lapse function N . This was interpreted as a breakdown of foliation invariance. One might then just as well abandon foliation invariance from the outset, and adopt a time-dependent Schrödinger equation with a specific choice of lapse N and a preferred time parameter t (as suggested in a de Broglie-Bohm context in refs. [23, 31, 42, 69]).

But the work of Pinto-Neto and Santini [70] appears to demonstrate that the above pilot-wave dynamics with the Wheeler-DeWitt equation is in fact well-posed. Pinto-Neto and Santini rewrite the dynamics of the de Broglie-Bohm trajectories in a classical Hamiltonian form. This is done by adding a term of the form Nq to the classical Hamiltonian density where

$$q = -\frac{1}{|\Psi|} G_{ijkl} \frac{\delta^2 |\Psi|}{\delta g_{ij} \delta g_{kl}} . \quad (20)$$

Given the guidance equation $p^{ij} = \delta S / \delta g_{ij}$ at an initial time, Hamilton's equations then generate the same de Broglie-Bohm trajectories as would be generated by the guidance equation applied at all times. By applying well-known theorems [71, 72, 73] it is then possible to show that, given (consistent) initial conditions on a spacelike slice, the resulting 4-geometry is independent of the choice of N and N^i . For $q \neq 0$, while the algebra of constraints is closed (when evaluated on the trajectories) it differs from the classical Dirac-Teitelboim algebra. Pinto-Neto and Santini conclude that, while the time evolution is consistent, in general it will form a spacetime with a non-Lorentzian structure (a degenerate 4-geometry) – unless q happens to vanish, in which case one recovers a classical evolution and a locally-Lorentzian spacetime. The breaking of the Dirac-Teitelboim algebra for $q \neq 0$ is interpreted as a breaking of local Lorentz invariance at the level of individual trajectories, caused by the nonlocality associated with $q \neq 0$.

However, if the pilot-wave dynamics of the Wheeler-DeWitt equation is indeed well-posed as a dynamical theory of a 3-geometry evolving in time, the question remains of how to connect the dynamics of single systems to the theory of a quantum equilibrium ensemble. This is usually trivial in pilot-wave theory, where the velocity field (3) is equal to the equilibrium probability current divided by the equilibrium probability density. But in the case of the Wheeler-DeWitt equation (18) there is no generally well-behaved candidate for

either of these quantities. If one attempts to straightforwardly interpret $|\Psi[g_{ij}]|^2$ as a probability density for quantum equilibrium, one is left with the difficulty of recovering a time dependence at the quantum level (where in general in de Broglie-Bohm theory the details of the trajectories are not observable in equilibrium). A common approach to solving this problem is to extract an appropriate degree of freedom $\tilde{t}[g_{ij}]$ from the 3-metric to play the role of time, so that $\Psi[g_{ij}]$ effectively becomes of the schematic form $\Psi[\tilde{g}_{ij}, \tilde{t}]$ where \tilde{g}_{ij} are the remaining metric variables. In quantum cosmology, for example, a popular choice of time variable is the scale factor a for an expanding universe. However, while this method certainly works in some cases, there seems to be no generally consistent way of extracting a well-behaved time function [14, 15, 16]. (For example, for a closed universe the ‘time’ a appears to stop and reverse at the point of maximum expansion, making it difficult to ensure that only one physical state is associated with each value of time.) As a result, there seems to be no generally well-behaved equilibrium current or time-dependent density for appropriate degrees of freedom \tilde{g}_{ij} .

Here we are touching on the notoriously controversial ‘problem of time’ in canonical quantum gravity. On one viewpoint, it might be asserted that our usual notion of time is meaningful only in certain emergent regimes, in which case it is to be expected that there is no generally well-defined time evolution. On another viewpoint, it might be suggested that the formalism is afflicted with a serious conceptual difficulty.⁵

On either view, a well-behaved equilibrium current and density may be generally said to emerge in those regimes where there is a Schrödinger-like equation $i\partial\Psi/\partial\tilde{t} = \hat{H}\Psi$ for a wave functional $\Psi[\tilde{g}_{ij}, \tilde{t}]$, with an effective Hamiltonian \hat{H} and time parameter \tilde{t} . If \hat{H} is given by a differential operator there will be an associated continuity equation [50]

$$\frac{\partial|\Psi|^2}{\partial\tilde{t}} + \int d^3x \frac{\delta J_{ij}}{\delta\tilde{g}_{ij}} = 0 , \quad (21)$$

where J_{ij} is a current. We may then define a de Broglie velocity field $\partial\tilde{g}_{ij}/\partial\tilde{t} = J_{ij}/|\Psi|^2$ and an ensemble of 3-geometries with an arbitrary distribution P of metrics \tilde{g}_{ij} will evolve according to

$$\frac{\partial P}{\partial\tilde{t}} + \int d^3x \frac{\delta}{\delta\tilde{g}_{ij}} \left(P \frac{\partial\tilde{g}_{ij}}{\partial\tilde{t}} \right) = 0 .$$

An ensemble with a distribution $P = |\Psi|^2$ at some initial time will then evolve into an ensemble with a distribution $P = |\Psi|^2$ at later times – the system will possess a quantum equilibrium state.

Outside of this ‘Schrödinger regime’, however, there appears to be no good reason to expect a quantum equilibrium state to exist. Of course, the mere fact

⁵Loop quantum gravity [9, 10, 11, 12] has technical advantages over the older metric representation being used here, but does not significantly improve the conceptual problem of time.

that one cannot apply the usual derivation of an equilibrium state does not by itself imply that there is no such state. But it is suggestive. It may simply be a mistake to assume that quantum gravity generally possesses a quantum equilibrium state described by a Born-like rule. We propose, then, that quantum equilibrium exists only in the Schrödinger-like regime. In pilot-wave theory, which is ultimately a dynamics of individual systems and not a dynamics of ensembles, it is in principle possible to consider this proposal in a conceptually coherent manner.

We may then expect quantum nonequilibrium to be generated by quantum-gravitational processes at the Planck scale. Schematically, consider an incoming state \mathcal{S}_{in} (with a wave functional and de Broglie-Bohm 3-geometry) that is accurately described by a Schrödinger regime, with a Schrödinger equation $i\partial\Psi_{\text{in}}/\partial\tilde{t}_{\text{in}} = \hat{H}_{\text{in}}\Psi_{\text{in}}$ and a quantum equilibrium state $|\Psi_{\text{in}}|^2$. Let us assume that the incoming state is indeed in equilibrium. The state could subsequently encounter interactions in the deep quantum-gravity regime, for which there is no well-defined Schrödinger equation and no well-defined quantum equilibrium state (a ‘non-Schrödinger regime’). One may end with an outgoing state \mathcal{S}_{out} that is again accurately described by a Schrödinger regime. However, the outgoing Schrödinger equation $i\partial\Psi_{\text{out}}/\partial\tilde{t}_{\text{out}} = \hat{H}_{\text{out}}\Psi_{\text{out}}$ and the outgoing quantum equilibrium state $|\Psi_{\text{out}}|^2$ may or may not coincide with their ingoing counterparts. In the absence of a single Schrödinger equation and associated probability current that describes the entire evolution from \mathcal{S}_{in} to \mathcal{S}_{out} , it is not possible to prove that an initial equilibrium state evolves to a final equilibrium state by integrating a single continuity equation. In such circumstances, there seems to be no obstruction to an incoming equilibrium state evolving into an outgoing *nonequilibrium* state. Such a transition, from equilibrium to nonequilibrium, could be established only in the context of a specific model. For example, such a scenario might be naturally applied to a bouncing model of quantum cosmology [74, 75]. More generally, our arguments suggest that quantum nonequilibrium could be generated for processes taking place at the Planck scale – such as the exit of trans-Planckian field modes from the Planck radius during inflation.

2.3 Regularisation of phase singularities in pilot-wave dynamics

Like classical general relativity, pilot-wave dynamics predicts its own demise. For standard Hamiltonians that are quadratic in the canonical momenta, the velocity field (5) generally diverges at nodes (where $\psi = 0$). Nodes are also known as ‘phase singularities’, where the phase $S = \text{Im} \ln \psi$ becomes ill-defined [76]. In a general n -dimensional configuration space, nodes form $(n - 2)$ -dimensional surfaces (as is clear from consideration of the simultaneous equations $\text{Re} \psi = 0$, $\text{Im} \psi = 0$ at fixed time t).

Thus for standard Hamiltonians pilot-wave theory breaks down at nodes. This elementary point is usually disregarded. In practice the divergence can be ignored because nodes form a set of measure zero (with respect to the standard

volume measure in configuration space). Even so, as a matter of principle the dynamics breaks down in these regions, signalling the possibility of new physics there.

While this divergence afflicts systems with Hamiltonians that are quadratic in the canonical momenta – for example nonrelativistic spinless particles – for some well-known systems the Hamiltonian is not of that form and there is no divergence. In particular, for a (high-energy) Dirac electron the one-body Dirac equation for a 4-component spinor ψ has a conserved current density $j^\mu = (j^0, j^i) = (\bar{\psi}\gamma^0\psi, \bar{\psi}\gamma^i\psi)$ (where the γ^μ are Dirac matrices) which may be used to define a natural velocity field $v^i = j^i/j^0$ and a de Broglie guidance equation $dx^i/dt = \bar{\psi}\gamma^i\psi/\bar{\psi}\gamma^0\psi$ [21]. (A similar construction may be given for the many-body case [52, 53, 54].) This velocity field is finite everywhere and indeed bounded by the speed of light c . It might then be suspected that the divergences could be an artifact of the low-energy, nonrelativistic theory. However, divergence at nodes is found in high-energy bosonic field theory, just as in the nonrelativistic particle case. For example, for a single (unentangled) mode \mathbf{k} of a free massless and real scalar field ϕ on Minkowski spacetime, if we write the Fourier components in terms of their real and imaginary parts, $\phi_{\mathbf{k}} \propto (q_{\mathbf{k}1} + iq_{\mathbf{k}2})$ (cf. Section 3), the wave function $\psi_{\mathbf{k}} = \psi_{\mathbf{k}}(q_{\mathbf{k}1}, q_{\mathbf{k}2}, t)$ of the mode satisfies [33, 34]

$$i\frac{\partial\psi_{\mathbf{k}}}{\partial t} = -\frac{1}{2}\left(\frac{\partial^2}{\partial q_{\mathbf{k}1}^2} + \frac{\partial^2}{\partial q_{\mathbf{k}2}^2}\right)\psi_{\mathbf{k}} + \frac{1}{2}k^2(q_{\mathbf{k}1}^2 + q_{\mathbf{k}2}^2)\psi_{\mathbf{k}},$$

and the de Broglie velocities for $q_{\mathbf{k}r}$ ($r = 1, 2$) are

$$\frac{dq_{\mathbf{k}r}}{dt} = \frac{\partial s_{\mathbf{k}}}{\partial q_{\mathbf{k}r}}$$

(with $\psi_{\mathbf{k}} = |\psi_{\mathbf{k}}|e^{is_{\mathbf{k}}}$). These equations are the same as in the pilot-wave theory of a nonrelativistic particle of unit mass in a simple harmonic oscillator potential in the $q_{\mathbf{k}1} - q_{\mathbf{k}2}$ plane, and the velocity field $\dot{q}_{\mathbf{k}r}$ exhibits the same divergence at nodes. Therefore, the physical motivation remains even in high-energy field theory.

If pilot-wave theory is taken seriously as a physical theory, the divergences must be removed or regularised by some mechanism. We suggest that their presence may be taken as a sign that new physics is needed at very short distances in configuration space (just as the presence of singularities in general relativity signals the need for new physics at very short distances in spacetime). Such new physics will presumably result in corrections to quantum mechanics. To find this new physics, one approach would be to seek new fundamental principles. Alternatively (or concurrently), one may develop simple phenomenological models and attempt to constrain them experimentally. The latter approach is followed here.

The need for regularisation in pilot-wave theory was briefly recognised in a paper by Bell (ref. [77], p. 138) where in a footnote it was remarked that the

velocity field (3) may be regularised by smearing the numerator j and denominator $|\psi|^2$ with a narrowly-peaked function (Bell suggested a Gaussian) in such a way that the smeared $|\psi|^2$ becomes the new equilibrium distribution.

While Bell did not write down any equations, his intentions are clear and easily reconstructed. Introducing a narrowly-peaked and positive-definite weighting function $\mu(q' - q)$ (for example a Gaussian) on configuration space, where $\int dq' \mu(q' - q) = 1$, we may define a regularised current

$$j(q, t)_{\text{reg}} = \int dq' \mu(q' - q) j(q', t) , \quad (22)$$

a regularised density

$$(|\psi(q, t)|^2)_{\text{reg}} = \int dq' \mu(q' - q) |\psi(q', t)|^2 , \quad (23)$$

and a regularised velocity field

$$v(q, t)_{\text{reg}} = \frac{j(q, t)_{\text{reg}}}{(|\psi(q, t)|^2)_{\text{reg}}} . \quad (24)$$

The latter may also be written as a ‘mean’

$$v(q, t)_{\text{reg}} = \frac{\int dq' \mu(q' - q) |\psi(q', t)|^2 v(q', t)}{\int dq' \mu(q' - q) |\psi(q', t)|^2} \quad (25)$$

of the unregularised field v with a weighting function $\mu|\psi|^2$.

We may then adopt the modified de Broglie equation of motion for the trajectories,

$$\frac{dq}{dt} = v(q, t)_{\text{reg}} , \quad (26)$$

together with the usual Schrödinger equation (1) for ψ . Equations (26) and (1) may be taken as the basic equations of a regularised pilot-wave dynamics. Assuming that $|\psi|^2$ vanishes only in regions of zero Lebesgue measure, $(|\psi|^2)_{\text{reg}}$ will be positive everywhere and the new velocity field v_{reg} will indeed be regular everywhere. The unregularised theory is recovered as the smearing function μ becomes arbitrarily narrow, $\mu(q' - q) \rightarrow \delta(q' - q)$.

Using (2), together with $\partial_q \mu(q' - q) = -\partial'_q \mu(q' - q)$ (where we write $\partial'_q = \partial_{q'}$), one finds that

$$\frac{\partial(|\psi|^2)_{\text{reg}}}{\partial t} + \partial_q \cdot j_{\text{reg}} = 0 \quad (27)$$

or

$$\frac{\partial(|\psi|^2)_{\text{reg}}}{\partial t} + \partial_q \cdot ((|\psi|^2)_{\text{reg}} v_{\text{reg}}) = 0 . \quad (28)$$

If we again consider an ensemble of systems with the same wave function $\psi(q, t)$, the time evolution of an arbitrary distribution $\rho(q, t)$ of configurations $q(t)$ will now obey the regularised continuity equation

$$\frac{\partial \rho}{\partial t} + \partial_q \cdot (\rho v_{\text{reg}}) = 0 . \quad (29)$$

Comparison of (28) and (29) shows that an initial distribution $\rho(q, t_i) = (|\psi(q, t_i)|^2)_{\text{reg}}$ at time t_i evolves into a final distribution

$$\rho(q, t) = (|\psi(q, t)|^2)_{\text{reg}} \quad (30)$$

at time t (where in general $(|\psi|^2)_{\text{reg}} \neq |\psi|^2$).

Thus the quantum equilibrium state is modified, or smeared, by the (narrow) regulator function μ , inducing deviations from the Born rule at small length-scales in configuration space. As in the unregularised theory, we may expect to find relaxation $\rho(q, t) \rightarrow (|\psi(q, t)|^2)_{\text{reg}}$ as quantified by the decrease of an H -function $H_{\text{reg}} = \int dq \rho \ln(\rho/(|\psi|^2)_{\text{reg}})$ (on a coarse-grained level).

The regularised equations are not supposed to be a candidate for a fundamental theory but instead are expected to provide an effective description of some deeper physics taking place at very short distances (much as in the analogous case of quantum field theory).

As we discuss in detail elsewhere, generally speaking it would be worth conducting experiments to probe the quantum probability distribution on small scales and to search for deviations from the Born rule in regions where the standard quantum-theoretical probability density approaches zero [78, 42]. In this paper we focus on the possible relevance to inflationary cosmology and physics at the Planck scale.

We have said that the regulator function μ should be regarded as an effective description of new physics at short distances in configuration space. In the above construction we assumed that μ was independent of time. But if we consider inflationary field modes that evolve from sub-Planckian to super-Planckian physical wavelengths, then because the modes make a transition between such different physical regimes it is plausible to suppose that during the transition μ could be time dependent. As we shall see, if the regulator function μ depends on time as the mode exits the Planck radius then quantum nonequilibrium will be generated from a prior equilibrium state – that is, the Born rule will become unstable at the Planck scale.

3 Pilot-wave dynamics and inflation

In Section 2.1 we formulated the pilot-wave theory of a massless (and minimally-coupled) real scalar field ϕ on a general globally-hyperbolic spacetime, with an assumed preferred foliation. Let us now consider the same field on an expanding flat space, with spacetime line element $d\tau^2 = dt^2 - a^2 d\mathbf{x}^2$ (where $a = a(t)$ is the scale factor and we take $c = 1$). This corresponds to a case with a uniform lapse function $N = 1$ and a 3-metric $g_{ij} = a^2 \delta_{ij}$ (with $g = a^6$). The dynamical equations (14) and (16) then become

$$i \frac{\partial \Psi}{\partial t} = \int d^3 \mathbf{x} \left(-\frac{1}{2a^3} \frac{\delta^2}{\delta \phi^2} + \frac{1}{2} a (\nabla \phi)^2 \right) \Psi \quad (31)$$

and

$$\frac{\partial \phi}{\partial t} = \frac{1}{a^3} \frac{\delta S}{\delta \phi} . \quad (32)$$

Working with Fourier components $\phi_{\mathbf{k}} = \frac{\sqrt{V}}{(2\pi)^{3/2}} (q_{\mathbf{k}1} + iq_{\mathbf{k}2})$ – with V a normalisation volume and $q_{\mathbf{k}r}$ ($r = 1, 2$) real variables – we have a Schrödinger equation

$$i \frac{\partial \Psi}{\partial t} = \sum_{\mathbf{k}r} \left(-\frac{1}{2a^3} \frac{\partial^2}{\partial q_{\mathbf{k}r}^2} + \frac{1}{2} a k^2 q_{\mathbf{k}r}^2 \right) \Psi \quad (33)$$

for $\Psi = \Psi[q_{\mathbf{k}r}, t]$ and de Broglie velocities

$$\frac{dq_{\mathbf{k}r}}{dt} = \frac{1}{a^3} \text{Im} \frac{1}{\Psi} \frac{\partial \Psi}{\partial q_{\mathbf{k}r}} = \frac{1}{a^3} \frac{\partial S}{\partial q_{\mathbf{k}r}} \quad (34)$$

(with $\Psi = |\Psi| e^{iS}$) for the evolving degrees of freedom $q_{\mathbf{k}r}$ [33, 34, 36]. The time evolution of an arbitrary distribution $P[q_{\mathbf{k}r}, t]$ is given by

$$\frac{\partial P}{\partial t} + \sum_{\mathbf{k}r} \frac{\partial}{\partial q_{\mathbf{k}r}} \left(P \frac{1}{a^3} \frac{\partial S}{\partial q_{\mathbf{k}r}} \right) = 0. \quad (35)$$

An unentangled mode \mathbf{k} has an independent dynamics with two degrees of freedom $q_{\mathbf{k}1}, q_{\mathbf{k}2}$. This has been used extensively to study cosmological relaxation for a radiation-dominated expansion with $a \propto t^{1/2}$ [33, 34, 36, 40, 41]. In the short-wavelength or sub-Hubble limit, we obtain the time evolution of a field mode on Minkowski spacetime and rapid relaxation takes place for a superposition of excited states; whereas for long (super-Hubble) wavelengths it is found that relaxation is retarded. If there was a radiation-dominated pre-inflationary era, we may expect incomplete relaxation at sufficiently long wavelengths – resulting in a large-scale power deficit in the inflationary spectrum [36, 40, 41].

Incomplete relaxation during a pre-inflationary era is one means by which nonequilibrium could exist in the inflationary spectrum. Another possibility – the subject of this paper – is that nonequilibrium is generated during inflation itself by novel gravitational effects at the Planck scale. This was suggested in ref. [36] (section IVB). As we have noted, trans-Planckian modes – that is, modes that originally had sub-Planckian physical wavelengths $\lambda_{\text{phys}} = a\lambda = a(2\pi/k)$ – may well make an observable contribution to the inflationary spectrum [5, 6], in which case inflation will allow us to probe physics at the Planck scale [7]. If quantum nonequilibrium is indeed generated at the Planck length l_{P} , an equilibrium mode with a physical wavelength $\lambda_{\text{phys}} < l_{\text{P}}$ in the early inflationary era would be driven out of equilibrium upon exiting the Planck radius (that is, when $\lambda_{\text{phys}} > l_{\text{P}}$) [36]. The inflaton field would then carry quantum nonequilibrium at short wavelengths (below a comoving cutoff).

However, a specific model of such a process has yet to be constructed. We shall do so here (Section 4). Our model of quantum instability will employ the results of refs. [33, 36], in which we calculated the de Broglie-Bohm trajectories for the inflaton perturbation ϕ in the inflationary (Bunch-Davies) vacuum. We recall the key results that will be needed to construct our model.

It is convenient to use conformal time η defined by $d\eta = dt/a$. For $a \propto e^{Ht}$ we have $\eta = -1/Ha$ and on an idealised de Sitter space η ranges over $(-\infty, 0)$.

The Bunch-Davies vacuum wave functional $\Psi[q_{\mathbf{k}r}, \eta]$ is a product $\prod_{\mathbf{k}r} \psi_{\mathbf{k}r}(q_{\mathbf{k}r}, \eta)$ of contracting Gaussian packets $\psi_{\mathbf{k}r}(q_{\mathbf{k}r}, \eta)$. Considering a single degree of freedom $q_{\mathbf{k}r}$ we may drop the index $\mathbf{k}r$. Writing the wave function $\psi = \psi(q, \eta)$ as $\psi = |\psi| e^{is}$, the conformal de Broglie velocity for the trajectory $q = q(\eta)$ is given by $dq/d\eta = adq/dt$ or (using (34))

$$\frac{dq}{d\eta} = (H\eta)^2 \frac{\partial s}{\partial q} . \quad (36)$$

As shown in ref. [36], the squared amplitude of ψ is a Gaussian

$$|\psi(q, \eta)|^2 = \frac{1}{\sqrt{2\pi}\Delta^2} e^{-q^2/2\Delta^2} \quad (37)$$

with a contracting width

$$\Delta(\eta) = \Delta(0) \sqrt{1 + k^2 \eta^2} , \quad (38)$$

where for convenience we write in terms of the asymptotic value

$$\Delta(0) = H/\sqrt{2k^3} . \quad (39)$$

For a calculation over a finite time interval (η_i, η_f) , we may write

$$\Delta(0) = \Delta(\eta_i)/\sqrt{1 + k^2 \eta_i^2} . \quad (40)$$

The phase of ψ is given by

$$s(q, \eta) = \frac{1}{2H^2} \frac{q^2}{\eta(\eta^2 + 1/k^2)} + h(\eta) \quad (41)$$

where $h(\eta) = \frac{1}{2} (-k\eta + \tan^{-1}(k\eta))$ is independent of q . Thus from (36) we have

$$\frac{dq}{d\eta} = \frac{q\eta}{\eta^2 + 1/k^2} . \quad (42)$$

The trajectories then take the simple form

$$q(\eta) = q(0) \sqrt{1 + k^2 \eta^2}$$

(where again for convenience we write in terms of the asymptotic value $q(0) = q(\eta_i)/\sqrt{1 + k^2 \eta_i^2}$).

The time evolution of an arbitrary distribution $\rho(q, \eta)$ is given by the general solution

$$\rho(q, \eta) = \frac{1}{\sqrt{1 + k^2 \eta^2}} \rho(q/\sqrt{1 + k^2 \eta^2}, 0) \quad (43)$$

of the continuity equation

$$\frac{\partial \rho}{\partial \eta} + \frac{\partial}{\partial q} (\rho q') = 0 \quad (44)$$

(where $q' \equiv dq/d\eta$). The distribution has a contracting width

$$D(\eta) = D(0)\sqrt{1 + k^2\eta^2} . \quad (45)$$

We obtain a homogeneous contraction of both ρ and $|\psi|^2$ by the same rescaling factor $1/\sqrt{1 + k^2\eta^2}$. Thus for each degree of freedom $q_{\mathbf{k}r}$ the width of ρ remains in a constant ratio with the width of $|\psi|^2$.

The ratio

$$\xi(k) \equiv \frac{\langle |\phi_{\mathbf{k}}|^2 \rangle}{\langle |\phi_{\mathbf{k}}|^2 \rangle_{\text{QT}}} \quad (46)$$

of the nonequilibrium variance $\langle |\phi_{\mathbf{k}}|^2 \rangle$ to the quantum-theoretical variance $\langle |\phi_{\mathbf{k}}|^2 \rangle_{\text{QT}}$ is then preserved in time. Relic nonequilibrium ($\xi \neq 1$) at the beginning of inflation will be preserved during the inflationary era and transferred to larger physical wavelengths λ_{phys} by the spatial expansion. By the same token, of course, initial equilibrium ($\xi = 1$) is also preserved in time. As it stands, the model does not allow nonequilibrium to be created from a prior equilibrium state. We shall now consider a modification of pilot-wave dynamics in which this is possible.

4 A model for quantum instability

In Section 2.3 we discussed the regularisation of phase singularities in pilot-wave theory. By smearing the quantum density and current with a narrow function $\mu(q' - q)$ we may define a regularised velocity field (24). This yields a modified equilibrium state (30) with deviations from the Born rule at small distances in configuration space. These deviations become arbitrarily small as $\mu(q' - q) \rightarrow \delta(q' - q)$. However, this construction assumes that the regulator function μ has no time dependence. But if μ is an effective description of new physics at short distances, and if we consider inflationary field modes that transition from the sub-Planckian regime ($\lambda_{\text{phys}} < l_{\text{P}}$) to the super-Planckian regime ($\lambda_{\text{phys}} > l_{\text{P}}$), then it appears reasonable to allow μ to be time dependent during the transition. As we shall now show, quantum nonequilibrium can then be generated from a prior equilibrium state. For a given regulator $\mu = \mu(q' - q, t)$ it is possible to calculate the time evolution away from equilibrium as the mode exits the Planck radius and so obtain an expression for the function $\xi(k)$ which quantifies deviations from the Born rule in the inflationary power spectrum.

4.1 Creation of nonequilibrium by a time-dependent regulator

For a general system with configuration q and time-dependent regulator $\mu(q' - q, t)$ (again with $\int dq' \mu(q' - q, t) = 1$) we may still define a regularised current

$$j(q, t)_{\text{reg}} = \int dq' \mu(q' - q, t) j(q', t) \quad (47)$$

and a regularised density

$$(|\psi(q, t)|^2)_{\text{reg}} = \int dq' \mu(q' - q, t) |\psi(q', t)|^2, \quad (48)$$

with a regularised velocity field $v(q, t)_{\text{reg}} = j(q, t)_{\text{reg}} / (|\psi(q, t)|^2)_{\text{reg}}$ or

$$v(q, t)_{\text{reg}} = \frac{\int dq' \mu(q' - q, t) |\psi(q', t)|^2 v(q', t)}{\int dq' \mu(q' - q, t) |\psi(q', t)|^2} \quad (49)$$

as before. We still have the de Broglie equation of motion $dq/dt = v(q, t)_{\text{reg}}$ for the trajectories and the Schrödinger equation (1) for ψ .

The time dependence of $\mu(q' - q, t)$ does however make one crucial difference: the regularised density $(|\psi(q, t)|^2)_{\text{reg}}$ is no longer an equilibrium state. To see this, note that an arbitrary distribution $\rho(q, t)$ still obeys the regularised continuity equation (29) whereas $(|\psi(q, t)|^2)_{\text{reg}}$ no longer obeys the (same) continuity equation (28). Instead, $(|\psi(q, t)|^2)_{\text{reg}}$ satisfies

$$\frac{\partial (|\psi|^2)_{\text{reg}}}{\partial t} + \partial_q \cdot ((|\psi|^2)_{\text{reg}} v_{\text{reg}}) = s \quad (50)$$

where the ‘source term’ s is given by

$$s(q, t) = \int dq' \frac{\partial \mu(q' - q, t)}{\partial t} |\psi(q', t)|^2. \quad (51)$$

(This follows from (2) with $\partial_q \mu(q' - q) = -\partial'_q \mu(q' - q)$.)

It now follows that an initial distribution $\rho(q, t_i) = (|\psi(q, t_i)|^2)_{\text{reg}}$ at time t_i in general evolves into a final distribution

$$\rho(q, t) \neq (|\psi(q, t)|^2)_{\text{reg}} \quad (52)$$

at time t , so that indeed $(|\psi(q, t)|^2)_{\text{reg}}$ is not an equilibrium state. This will be confirmed below for a simple example. In general, from (29) and (50) it follows that the regularised ratio

$$f_{\text{reg}} \equiv \frac{\rho}{(|\psi|^2)_{\text{reg}}}$$

satisfies

$$\frac{df_{\text{reg}}}{dt} = -u f_{\text{reg}} \quad (53)$$

where

$$u \equiv \frac{s}{(|\psi|^2)_{\text{reg}}}$$

and now $d/dt = \partial/\partial t + v_{\text{reg}} \cdot \partial_q$. Integrating (53) along a trajectory from an initial point q_i at time t_i to a final point q_f at time t_f we have

$$f_{\text{reg}}(q_f, t_f) = f_{\text{reg}}(q_i, t_i) \cdot \exp \left(- \int_{\text{traj}} dt u(q(t), t) \right).$$

In general, $\int_{\text{traj}} u dt \neq 0$ and $f_{\text{reg}}(q_f, t_f) \neq f_{\text{reg}}(q_i, t_i)$.

We may then consider the following type of scenario. At times $t < t_i$ and $t > t_f$ the regulator μ is time independent and $s = 0$. At these times we will have a regularised equilibrium distribution $(|\psi|^2)_{\text{reg}}$. Should μ be time dependent during the interval (t_i, t_f) , then an incoming equilibrium state $\rho = (|\psi|^2)_{\text{reg}}$ will evolve into an outgoing *nonequilibrium* state $\rho \neq (|\psi|^2)_{\text{reg}}$.

Such a scenario may be applied to a case where for $t < t_i$ and $t > t_f$ the regularisation may be neglected, with $\mu(q' - q, t) = \delta(q' - q)$ (approximately). At these times we will have the standard Born-rule equilibrium distribution $(|\psi|^2)_{\text{reg}} = |\psi|^2$. If μ is time dependent during the interval (t_i, t_f) , an incoming Born-rule distribution $\rho = |\psi|^2$ will evolve into an outgoing non-Born-rule distribution $\rho \neq |\psi|^2$.

4.2 Calculation of the nonequilibrium function $\xi(k)$

We may now apply these considerations to an inflationary field mode, yielding a model in which quantum nonequilibrium is created during inflation as trans-Planckian modes exit the Planck radius. For this purpose it will be convenient to use conformal time η . As we saw in Section 3, the Bunch-Davies vacuum wave function $\psi(q, \eta)$ is a contracting Gaussian. While this wave function does not possess nodes ($\psi \neq 0$ for all finite q), even so the generic presence of nodes for arbitrary wave functions – generally superpositions of the vacuum with excited states – implies a need for regularisation at short distances in configuration space. As we discussed in Section 2.3, such regularisation may be viewed as an effective description of new physics. We will assume this new physics to be present even if ψ happens to be free of nodes.

We may now reconsider the results for the inflationary vacuum (summarised in Section 3) including the presence of a time-dependent regulator $\mu(q' - q, \eta)$. For definiteness we consider a simple example. We take

$$\mu(q, \eta) = \delta_\alpha(q) = \frac{1}{\sqrt{2\pi\alpha^2}} e^{-q^2/2\alpha^2}, \quad (54)$$

where $\delta_\alpha(q)$ is a regularised delta-function of time-dependent width $\alpha = \alpha(\eta)$ (with $\alpha \geq 0$ for all η). Note that $\delta_\alpha(q) \rightarrow \delta(q)$ as $\alpha \rightarrow 0$. We take

$$\alpha(\eta_i) = \alpha(\eta_f) = 0, \quad (55)$$

so that the regulator is switched off at the initial and final times η_i and η_f . Strictly speaking, it would be more realistic to take $\alpha(\eta_i)$ and $\alpha(\eta_f)$ to be very small but non-zero, so that the regulator is negligible initially and finally. We assume that $\alpha = \alpha(\eta)$ is non-negligible and time dependent during the interval (η_i, η_f) , so that an incoming equilibrium state evolves into an outgoing nonequilibrium state.

We assume that the interval (η_i, η_f) straddles the time at which the mode exits the Planck radius. Thus, according to this model, regularisation is constant and negligible in the far sub-Planckian and far super-Planckian regimes but regularisation is non-negligible and time dependent during Planck radius crossing.

As a consequence, our model then describes the creation of nonequilibrium for trans-Planckian modes emerging from the Planck radius.

Applying the Gaussian regulator (54) to the Bunch-Davies vacuum, from (48) we find a regularised density

$$(|\psi|^2)_{\text{reg}} = \frac{1}{\sqrt{2\pi(\Delta^2 + \alpha^2)}} e^{-q^2/2(\Delta^2 + \alpha^2)} . \quad (56)$$

This is still a Gaussian packet but with a modified width $\sqrt{\Delta^2 + \alpha^2}$. From (47) we also find a regularised current

$$j_{\text{reg}} = \frac{\Delta^2}{\Delta^2 + \alpha^2} \frac{q\eta}{\eta^2 + 1/k^2} (|\psi|^2)_{\text{reg}} . \quad (57)$$

The regularised de Broglie velocity is then given by

$$v(q, \eta)_{\text{reg}} \equiv j_{\text{reg}} / (|\psi|^2)_{\text{reg}} = \frac{\Delta^2}{\Delta^2 + \alpha^2} \frac{q\eta}{\eta^2 + 1/k^2} = \frac{\Delta^2}{\Delta^2 + \alpha^2} v(q, \eta) , \quad (58)$$

where $v(q, \eta)$ is the unregularised velocity field (42). It is convenient to define a function $g^2(\eta)$ by

$$g^2 = (1/k^2)(1 + \alpha^2/\Delta_0^2) , \quad (59)$$

where (from (38)) $\Delta_0^2 = \Delta^2/(1 + k^2\eta^2)$ is the unregularised equilibrium variance at $\eta = 0$. From (55) we have $g^2(\eta_i) = g^2(\eta_f) = 1/k^2$. Since

$$\frac{\Delta^2}{\Delta^2 + \alpha^2} = \frac{\Delta_0^2(1 + k^2\eta^2)}{\Delta_0^2(1 + k^2\eta^2) + \alpha^2} = \frac{\eta^2 + 1/k^2}{\eta^2 + g^2} , \quad (60)$$

we then have a modified de Broglie equation of motion

$$\frac{dq}{d\eta} = v(q, \eta)_{\text{reg}} = \frac{q\eta}{\eta^2 + g^2} \quad (61)$$

for the evolving degree of freedom $q = q(\eta)$.

For any function $g^2(\eta)$, our initial equilibrium Gaussian $\rho(q, \eta_i) = |\psi(q, \eta_i)|^2$ of width Δ_i (and zero mean) evolves into a final nonequilibrium Gaussian of width

$$D_f = X_{fi} \Delta_i \quad (62)$$

(and zero mean) where

$$X_{fi} \equiv \exp \left(\int_{\eta_i}^{\eta_f} d\eta \frac{\eta}{\eta^2 + g^2} \right) . \quad (63)$$

To see this, consider a trajectory $q = q(\eta)$ that begins at $q_i = q(\eta_i)$ and ends at $q_f = q(\eta_f)$. A simple integration of (61) yields

$$q_f = q_i X_{fi} . \quad (64)$$

Furthermore, for the evolving distribution $\rho(q, \eta)$ we necessarily have

$$\rho(q_f, \eta_f) dq_f = \rho(q_i, \eta_i) dq_i$$

(since trajectories beginning in a neighbourhood dq_i of q_i end in a neighbourhood dq_f of q_f). Because $\rho(q_i, \eta_i) = |\psi(q_i, \eta_i)|^2$ (by assumption), from (64) we then have the final distribution

$$\rho(q_f, \eta_f) = \frac{1}{X_{fi}} |\psi(q_f/X_{fi}, \eta_i)|^2 . \quad (65)$$

This is indeed a Gaussian of width $D_f = X_{fi} \Delta_i$ and zero mean.

It is also straightforward to show that in this model we always obtain a final super-quantum width

$$D_f > \Delta_f . \quad (66)$$

To see this, note that from (38) the final equilibrium width Δ_f may be written as

$$\Delta_f = \Delta_i \sqrt{\frac{\eta_f^2 + 1/k^2}{\eta_i^2 + 1/k^2}} . \quad (67)$$

From (59) we have $g^2 > 1/k^2$ in the interval (η_i, η_f) (assuming that α does not always vanish). From (63) we then have (noting that $\eta < 0$)

$$X_{fi} > \exp \left(\int_{\eta_i}^{\eta_f} d\eta \frac{\eta}{\eta^2 + 1/k^2} \right) = \sqrt{\frac{\eta_f^2 + 1/k^2}{\eta_i^2 + 1/k^2}} \quad (68)$$

and so indeed $D_f > \Delta_f$. Thus, according to this model, the time-dependent regulator always generates a power excess in the primordial perturbations (in the relevant region of k -space).

From (62) and (67), at the final time η_f we have a nonequilibrium function

$$\xi(k) \equiv \frac{D_f^2}{\Delta_f^2} = \left(\frac{\eta_i^2 + 1/k^2}{\eta_f^2 + 1/k^2} \right) X_{fi}^2 , \quad (69)$$

with X_{fi} given by (63). For any given regularisation – specified by $\alpha(\eta)$, or equivalently by $g^2(\eta)$ – we may calculate X_{fi} and so find $\xi(k)$.

For example, let us consider a simple quadratic form

$$g^2 = (a - 1)\eta^2 + b\eta + c \quad (70)$$

with constants a , b and c chosen so that $g^2(\eta_i) = g^2(\eta_f) = 1/k^2$ (since $\alpha(\eta_i) = \alpha(\eta_f) = 0$). For the case $b^2 < 4ac$ we find

$$\xi(k) = \left(\frac{\Delta_f}{\Delta_i} \right)^{2(1-a)/a} \exp(\gamma_f - \gamma_i) , \quad (71)$$

where

$$\gamma \equiv -\frac{2b}{a} \frac{1}{\sqrt{4ac - b^2}} \tan^{-1} \left(\frac{2a\eta + b}{\sqrt{4ac - b^2}} \right) . \quad (72)$$

The constants a , b , c may equally be written in terms of η_i , η_f and a constant d :

$$a = 1 - d, \quad b = (\eta_i + \eta_f)d, \quad c = 1/k^2 - \eta_i\eta_f d.$$

The dependence on k is contained in c .

These illustrative examples serve as a starting point. One could of course consider other choices for the regulator function and explore the extent to which the results depend on the choice made.

5 Trans-Planckian phenomenology and the CMB

During the inflationary era an inflaton perturbation $\phi_{\mathbf{k}}$ generates a curvature perturbation $\mathcal{R}_{\mathbf{k}} \propto \phi_{\mathbf{k}}$ (after the mode exits the Hubble radius), which in turn generates the CMB angular power spectrum [1]

$$C_l = \frac{1}{2\pi^2} \int_0^\infty \frac{dk}{k} \mathcal{T}^2(k, l) \mathcal{P}_{\mathcal{R}}(k), \quad (73)$$

where $\mathcal{T}(k, l)$ is the transfer function and

$$\mathcal{P}_{\mathcal{R}}(k) \equiv \frac{4\pi k^3}{V} \langle |\mathcal{R}_{\mathbf{k}}|^2 \rangle \quad (74)$$

is the primordial power spectrum. From (46) the nonequilibrium power spectrum may be written as

$$\mathcal{P}_{\mathcal{R}}(k) = \mathcal{P}_{\mathcal{R}}^{\text{QT}}(k) \xi(k), \quad (75)$$

where $\mathcal{P}_{\mathcal{R}}^{\text{QT}}(k)$ is the quantum-theoretical or equilibrium power spectrum. Measurements of C_l may be used to set bounds on the deviation of $\xi(k)$ from 1 [36].

Primordial quantum nonequilibrium is quantified by the function $\xi(k)$. Extensive numerical studies of relaxation during a pre-inflationary era indicate that $\xi(k)$ will take the form of an inverse-tangent – with a power deficit $\xi < 1$ at small k and with $\xi \simeq 1$ at large k – where the deficit is caused by incomplete relaxation at long wavelengths [40, 41]. A large-scale power deficit has been found in data gathered by the *Planck* mission [49]. The magnitude and location of the deficit are broadly consistent with pre-inflationary relaxation suppression [41]. But whether or not the predicted function $\xi(k)$ is supported by the data remains to be seen [79].

In this paper we are concerned with a different scenario. Instead of considering relic nonequilibrium from earlier times, we are exploring the possibility that nonequilibrium is created during inflation by novel effects at the Planck scale. Nonequilibrium would then be expected to set in at wavenumbers k larger than some critical value k_c or at wavelengths smaller than $\lambda_c = 2\pi/k_c = \lambda_{\text{max}}$ – where modes of wavelength longer than λ_{max} were never sub-Planckian during the inflationary phase (see Figure 1).

For a region of k -space that is potentially subject to trans-Planckian effects, we may attempt to predict features of the nonequilibrium function $\xi(k)$. Our example of a time-dependent regulator leads to the form (71) for $\xi(k)$. This result depends on our simple choice – defined by (54) and (70) – for the regulator. At present we have no theoretical foundation for the regulator, which we have introduced as an effective description of new and unknown physics at the Planck scale. Therefore any test of the predicted modification of the power spectrum by the function $\xi(k)$ may be seen as constraining the regulator function. If one does adopt our simple choice of regulator, the parameters appearing in the resulting expression (71) for $\xi(k)$ are of course unknown but in principle the general form of this function could be supported (or not) by the data. This would require performing a best-fit to the data, with the parameters in (71) freely varying, to find out if the fit is statistically significant or not. This is a matter for future work.

We have noted that our model with a time-dependent regulator can generate only a power excess ($\xi > 1$) and never a deficit ($\xi < 1$). It may then seem that this model could never account for the long-wavelength deficit reported by the *Planck* mission [49]. However, if there is a general power excess below a critical wavelength λ_c , then it could happen that when the measured CMB power spectrum is normalised it will appear *as if* there were a power deficit above the same critical wavelength λ_c . Should λ_c be comparable to the Hubble radius H_0^{-1} today, our model could then predict an effective deficit in the observed region.

To delineate the region of k -space that is potentially subject to trans-Planckian effects – specifically, to correction of the power spectrum by the factor $\xi(k)$ – we may consider the following simple estimates.

If inflation begins at a time t_{begin} and ends at a time t_{end} , then with an inflationary Hubble parameter H the number of e-folds will be $N = H(t_{\text{end}} - t_{\text{begin}})$. The relevant range of k – where trans-Planckian effects can occur in the inflationary spectrum – is determined by maximum and minimum wavelengths λ_{max} and λ_{min} , where modes with comoving wavelengths larger than λ_{max} were never sub-Planckian during inflation while modes with comoving wavelengths smaller than λ_{min} do not exit the Hubble radius before inflation ends (see Figure 1). Thus

$$a_{\text{begin}}\lambda_{\text{max}} \simeq l_{\text{P}} \quad (76)$$

and

$$a_{\text{end}}\lambda_{\text{min}} \simeq H^{-1} . \quad (77)$$

We then have relevant wave numbers k in the range $(2\pi/\lambda_{\text{max}}, 2\pi/\lambda_{\text{min}})$.

In practice λ_{min} will be so small that we may as well take it to be zero (see below). Thus the effects can set in at an ultraviolet cutoff $\lambda_c = \lambda_{\text{max}}$ or

$$\lambda_c \simeq l_{\text{P}}/a_{\text{begin}} . \quad (78)$$

Modes with comoving wavelengths $\lambda > \lambda_c$ were never sub-Planckian during the inflationary phase and we can assume they are in equilibrium (even if they were

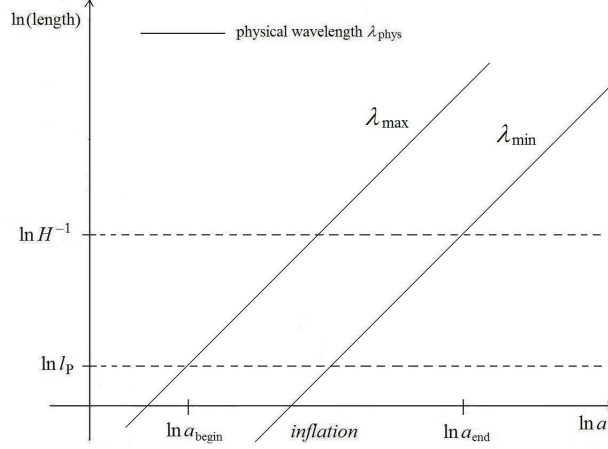


Figure 1: Physical wavelengths of trans-Planckian field modes during inflation.

sub-Planckian during pre-inflation, we may assume they relax to equilibrium during pre-inflation after they exit l_P).

We have $a_{\text{begin}} = a_{\text{end}} e^{-N}$ and so

$$\lambda_c \simeq l_P e^N / a_{\text{end}} .$$

If we neglect the expansion that takes place during the transition from inflation to post-inflation, we can write $a_{\text{end}}/a_0 \simeq T_0/T_{\text{end}}$ (where T_0 is the temperature today and T_{end} is the temperature at which inflation ends). Taking $a_0 = 1$ it follows that

$$\lambda_c \simeq l_P e^N (T_{\text{end}}/T_0) . \quad (79)$$

Using $l_P \simeq 10^{-33}$ cm and writing $(1 \text{ cm}) \simeq H_0^{-1} e^{-65}$ (where $H_0^{-1} \simeq 10^{28}$ cm), we then have

$$\lambda_c \simeq 10^{-33} H_0^{-1} e^{(N-65)} (T_{\text{end}}/T_0) . \quad (80)$$

Inflation can solve the horizon and flatness problems with a minimum number of e-folds that is usually estimated to lie in the range $N_{\text{min}} \simeq 60 - 70$ [4]. The actual number N of e-folds could of course be much larger than N_{min} [80]. The ‘reheating temperature’ T_{end} depends on details of the reheating process such as the inflaton decay rate [4, 81]. Constraints from CMB data yield lower bounds on T_{end} in the range 390 GeV – 890 TeV (depending on the inflationary model) [82]. Denoting the temperature at the beginning of inflation by T_{begin} , estimates for $T_{\text{end}}/T_{\text{begin}}$ can range from $T_{\text{end}}/T_{\text{begin}} \sim 1$ to $T_{\text{end}}/T_{\text{begin}} \ll 1$. We may reasonably take T_{begin} to be of the same order of magnitude as the energy scale $H \sim 10^{16}$ GeV $\sim 10^{-3} T_P$ associated with the inflationary phase.

We may write (80) as

$$\lambda_c \simeq 10^{-33} H_0^{-1} e^{(N-65)} (T_{\text{end}}/1 \text{ TeV}) (1 \text{ TeV}/T_0) .$$

Using $T_0 \sim 10^{-4}$ eV we then have

$$\lambda_c \simeq 10^{-17} H_0^{-1} e^{(N-65)} (T_{\text{end}}/1 \text{ TeV}) . \quad (81)$$

As an illustrative example, the estimate (81) yields an order of magnitude $\lambda_c \sim H_0^{-1}$ if

$$e^{(N-65)} (T_{\text{end}}/1 \text{ TeV}) \sim 10^{17} . \quad (82)$$

This is consistent with the allowed parameter space. For example, we could have $T_{\text{end}} \sim 10^{-3} T_{\text{P}}$ or $T_{\text{end}}/1 \text{ TeV} \sim 10^{13}$ together with $N \sim 75$. To have much less than the ‘maximal’ reheating temperature requires a larger number of e-folds. For example, to have $T_{\text{end}}/1 \text{ TeV} \sim 1$ we would need $N \sim 105$.

As for λ_{min} , from (77) and using $a_{\text{end}} = a_{\text{begin}} e^N$ we may write

$$\lambda_{\text{min}} \simeq (H^{-1}/l_{\text{P}}) e^{-N} (l_{\text{P}}/a_{\text{begin}}) \simeq (H^{-1}/l_{\text{P}}) e^{-N} \lambda_{\text{max}} .$$

Thus λ_{min} is exponentially smaller than λ_{max} . From (79) we have $\lambda_{\text{max}} \simeq l_{\text{P}} e^N (T_{\text{end}}/T_0)$ and so

$$\lambda_{\text{min}} \simeq H^{-1} (T_{\text{end}}/T_0) . \quad (83)$$

For an inflationary energy scale $H \sim 10^{16}$ GeV we have $H^{-1} \sim \hbar c/(10^{16} \text{ GeV}) \simeq 10^{-30}$ cm. As for the ratio T_{end}/T_0 , taking a maximal value $T_{\text{end}} \lesssim 10^{16}$ GeV (with $T_{\text{end}}/T_{\text{begin}} \lesssim 1$) we have $\lambda_{\text{min}} \lesssim 10^{-1}$ cm. This is indeed completely negligible and we may as well take $\lambda_{\text{min}} = 0$.

6 Discussion and conclusion

It is remarkable that trans-Planckian physics may be observable in the CMB, enabling the above theoretical proposals to be constrained by experiment. Trans-Planckian effects from more standard corrections to quantum field theory (standard in the sense of remaining within the quantum formalism with the usual Born rule) have been discussed by a number of authors. It appears that oscillations in the primordial power spectrum are a generic prediction of such models, though whether such features exist in the data remains a topic of research [7]. If one is willing to entertain the possibility of trans-Planckian corrections to the Born rule, as suggested here, it will be essential to find characteristic signatures that would enable such effects to be distinguished from others. Our model with a time-dependent regulator yields a power excess $\xi > 1$. In our example ξ was found to be given by the expression (71) as a function of k , though this result depends on our simple choice of regulator for which we have as yet no theoretical foundation. It may be hoped that deeper theoretical developments will lead to firmer predictions, and that further analysis of the data (for example best-fitting to power spectra corrected by factors of the form (71)) will provide more detailed constraints on the kind of model proposed here.

Of the three arguments provided in Section 2 for quantum instability at the Planck scale, we have focussed on a model with a time-dependent regularisation

of pilot-wave dynamics. Elsewhere [75] we study quantum-gravitational models for which there is no stable equilibrium state in the deep quantum-gravity regime, as outlined in Section 2.2. It remains to be seen what predictions could emerge from such models.

It was argued by Weiss [83] that, in a quantum field theory with an ultra-violet cutoff at a fixed physical lengthscale, on an expanding background the number of field modes required to describe the physics will increase with time. In effect, new degrees of freedom are born as the universe expands. According to inflationary cosmology, these ‘new modes’ could make an observable contribution to the CMB spectrum [7]. From a pilot-wave perspective, the Born rule is a contingency and so there seems to be no particular reason for why new modes should begin in a state of quantum equilibrium. One might attempt to derive an estimate for the magnitude of the nonequilibrium ratio ξ for newly-born modes from an information-theoretic argument. The ‘hidden-variable entropy’ $S_{\text{hv}}(k)$ of a field mode k may be taken to be minus the H -function (10) for the mode [36]. As a mode exits the Planck radius, it might be said that a new degree of freedom is being created. Without attempting a proper justification, we could assume that the hidden-variable entropy of these new degrees of freedom will be given by $S_{\text{hv}} \sim -\ln 2$. One could think of this (negative) entropy as being generated so as to ‘compensate’ for the creation of a new degree of freedom.⁶ For Gaussian packets ρ and $|\psi|^2$ of respective widths D and Δ , we find $S_{\text{hv}} = -H = \frac{1}{2}(1 - \xi + \ln \xi)$ where $\xi = D^2/\Delta^2$. If we indeed assume that $S_{\text{hv}} \sim -\ln 2$, we have (in the region of k -space where such effects could be relevant) $1 - \xi + \ln \xi \sim -2\ln 2$. This yields two solutions, $\xi \sim 0.1$ and $\xi \sim 3.7$, with respective sub-quantum and super-quantum widths. Something more is needed to select one value over the other. Whether a rigorous argument can be constructed along these lines remains to be seen.

In the context of pilot-wave theory, it is natural to ask why our universe today is (at least to a good approximation) in a state of quantum equilibrium while at the same time being in a state that is far from thermal equilibrium. According to our current understanding, we observe thermal nonequilibrium today because in the early universe gravitation amplified the small inhomogeneities in temperature and energy density, leading to the formation of large-scale structure [84]. Were it not for this peculiarity of gravitation, our universe would now be in a state of global thermal equilibrium. In contrast, we do observe global quantum equilibrium today: all systems we have access to have been found to obey the Born rule (to high accuracy). It might then seem that there can be no quantum analogue of the gravitational amplification of thermal fluctuations. However, according to our proposals, there can be circumstances – albeit at the Planck scale – in which gravitation drives systems away from quantum equilibrium.⁷ If such effects do exist in the very early universe, quantum nonequilibrium will be present at very early times – even if the universe began in a state of quantum

⁶The relationship between S_{hv} and the usual von Neumann entropy is not fully understood; for a discussion see refs. [56, 42].

⁷In refs. [56, 33] it was suggested that such effects could exist at macroscopic scales for entangled states straddling the event horizon of an evaporating black hole.

equilibrium.

The proposals made in this paper arguably strengthen the parallels between quantum and thermal fluctuations which have been noted by some authors in a gravitational context. In particular, it has been argued that in the presence of gravitation there is no invariant distinction between quantum and thermal fluctuations [85, 86]. After several decades, possible deep connections between quantum theory, gravitation and statistical physics remain tantalising.

If inflationary cosmology does indeed open an empirical window onto physics at the Planck scale and beyond, then as well as considering the numerous modifications of high-energy physics that have been proposed in the literature we should take into account the possibility that quantum theory itself could break down in such extreme conditions. It would therefore also be of interest to explore inflationary collapse models [87, 88, 89, 90] in a trans-Planckian context. We have advanced arguments suggesting that quantum mechanics is unstable at the Planck scale, and we have provided an illustrative model that makes use of the contingent status of the Born rule in the de Broglie-Bohm formulation of quantum theory. We suggest that these considerations are due for further development, and that the Born rule should take its place alongside other basic features of modern physics that may be questioned in the deep high-energy regime potentially probed by inflationary cosmology.

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